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1990 J. Phys. A: Math. Gen. 23 L1019

(http://iopscience.iop.org/0305-4470/23/19/001)

J. Phys. A: Math. Gen. 23 (1990) L1019–L1023. Printed in the UK

LETTER TO THE EDITOR

q-analogues of the parabose and parafermi oscillators and representations of quantum algebras

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Received 6 July 1990

Abstract. Deformations of the one-dimensional parabose oscillators are presented. They are shown to provide Fock representations of the quantum $sp_q(2)$ algebra and $osp_q(1,2)$ superalgebra. The representations of $su_q(2)$ are used to define the q-analogues of the parafermi oscillators.

Quantum groups and algebras have been shown to arise in many problems of current physical and mathematical interest. Much effort is now being devoted to the construction of their representations and recently many realizations have been usefully devised using q-deformations of boson and fermion operators [1–14]. In the present letter, we introduce q-analogues of the (one-dimensional) parabose and parafermi oscillators and discuss the representations of the quantum algebras $sp_q(2)$, $osp_q(1,2)$ and $su_q(2)$ that they entail.

Introduce a Hilbert space whose basis vectors $|n\rangle$, $n \in \mathbb{N}$, are eigenvectors of a Hermitian operator N and let b and $b^\dagger$ be a pair of Hermitian-conjugate operators acting on that space and satisfying

$$[N, b] = -b \qquad [N, b^\dagger] = b^\dagger. \tag{1}$$

We shall consider first bose-like oscillators and shall therefore assume that the spectrum of N is of the form

$$N_n = n + \frac{p}{2} \qquad n = 0, 1, 2, \ldots, p \geqslant 0. \tag{2}$$

Consistently with (1), we can take the operators b and $b^\dagger$ to have the following non-vanishing matrix elements:

$$b_{n,n+1} = b^\dagger_{n+1,n} = \begin{cases} ([n+p]_q)^{1/2} & \text{for } n \text{ even} \\ ([n+1]_q)^{1/2} & \text{for } n \text{ odd.} \end{cases} \tag{3}$$

We are using the notation

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{4}$$

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with $q \in \mathbb{C}$, $q^2 \neq 1$. Note that $\lim_{q\to 1} [x]_q = x$. Relation (3) defines a q-deformation of the parabose oscillator; the matrix elements of the undeformed parabose annihilation and creation operators [15] are recovered in the limit $q \to 1$, where $N \to \frac{1}{2}(bb^\dagger + b^\dagger b)$. The integer $p = 1, 2, \ldots$ is referred to as the order of paraquantization. When $p = 1$, the operators b and $b^\dagger$ given in (3) satisfy the following commutation relation

$$[b, b^\dagger] = [1/2]_q [2]_{q^N} = \frac{q^N + q^{-N}}{q^{1/2} + q^{-1/2}} \tag{5}$$

which clearly reduces to $[b, b^\dagger] = 1$ as q goes to 1. In this case, however, it has been more customary to use instead of (5), and equivalently, the relation

$$bb^\dagger - qb^\dagger b = q^{-N}. \tag{6}$$

As is easy to check, this equation is also verified when definition (3) is used.

The q-analogues of the second-order ($p = 2$) parabose annihilation and creation operators are seen to satisfy the commutation relation

$$b^2 b^\dagger - b^\dagger b^2 = [2]_{q^{(N+\pi(N))}} b \tag{7}$$

(plus the corresponding Hermitian conjugate relation), where

$$\pi(N) = \tfrac{1}{2}(1 + (-1)^N). \tag{8}$$

Again, when $q \to 1$, (7) reduces to $b^2 b^\dagger - b^\dagger b^2 = 2b$, typical of second-order (classical) parabose operators [15]. For higher orders, the commutation relations become more and more involved. In deriving relations like (5) and (7), the following formulae prove useful:

$$\begin{aligned} &[2k + l]_q + [l]_q = [k + l]_q [2]_{q^k} \\ &[2k + l]_q - [l]_q = [2]_{q^{k+l}} [k]_q. \end{aligned} \tag{9}$$

Using (3), one can now construct representations of the quantized universal algebra $\mathrm{sp}_q(2)$ of $\mathrm{sp}(2)$ on the Fock space $\mathscr{H}^{(p)}$ built with one q-parabose oscillator of (arbitrary) order p. The quantum algebra $\mathrm{sp}_q(2)$ is generated by three elements e, f and h which satisfy the defining relations [12, 13]

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = [h]_{q^2}. \tag{10}$$

It possesses the following Casimir operator

$$C = ([(h - 1)/2]_{q^2})^2 - ef. \tag{11}$$

In terms of the q-parabose operators defined in (3), this algebra is realized by taking

$$e = \frac{1}{[2]_q} (b^\dagger)^2 \qquad f = \frac{1}{[2]_q} b^2 \qquad h = N. \tag{12}$$

It is straightforward to check that (10) are satisfied under this identification. For a given p, this representation of $\mathrm{sp}_q(2)$ on $\mathscr{H}^{(p)}$ decomposes into two irreducible components, with the invariant subspaces formed out of the states with an even or an odd number of q-oscillator excitations. Correspondingly, the Casimir operator is found to take the values

$$C = \begin{cases} ([(p - 2)/4]_{q^2})^2 & \text{for } n \text{ even} \\ ([p/4]_{q^2})^2 & \text{for } n \text{ odd.} \end{cases} \tag{13}$$

In the undeformed limit ($q \to 1$), these two components become irreducible representations of the ordinary Lie algebra sp(2). Irrespective of the order p, it is known [16] that they can be embedded in a single irreducible representation of the ordinary superalgebra osp(1, 2), which is generated by the linears and bilinears in the oscillator annihilation and creation operators. In the quantum case, the $p = 1$ irreducible representation of $\mathrm{sp}_q(2)$ described above can also be embedded [8, 9] into an irreducible representation of the quantum superalgebra $\mathrm{osp}_q(1, 2)$. We shall now examine how this extends to higher-order cases.

The quantum superalgebra $\mathrm{osp}_q(1, 2)$ is generated by three elements E, F and H subjected to the relations [17-20]

$$[H, E] = 2E \qquad [H, F] = -2F \qquad \{E, F\} \equiv EF + FE = [H]_q. \tag{14}$$

The q-oscillator realizations of this algebra are best described by modifying b and $b^\dagger$ of (3) into the operators

$$\tilde{b} = \sqrt{\mathscr{F}_q(p, N)}\, b \qquad \tilde{b}^\dagger = \sqrt{\mathscr{F}_q(p, N-1)}\, b^\dagger \tag{15}$$

with $\mathscr{F}_q(p, N)$ defined through

$$F_q(p, N)|n\rangle = \begin{cases} [2]_{q^{(n+1)}}/[2]_q & \text{for } n \text{ even} \\ [2]_{q^{(p+n)}}/[2]_q & \text{for } n \text{ odd.} \end{cases} \tag{16}$$

Note that $\mathscr{F}_q(p, N) \to 1$ as $q \to 1$. The operators $\tilde{b}$ and $\tilde{b}^\dagger$ therefore also provide a q-deformation of the parabose annihilation and creation operators. The q-commutation relations take a simpler form when these generators are used. For $p = 1$, one has

$$[\tilde{b}, \tilde{b}^\dagger] = [1/2]_{q^2}[2]_{q^{2N}} \tag{17}$$

which coincides with (5) after the redefinition $q^2 \to q$, while for $p = 2$, one finds

$$\tilde{b}^2\tilde{b}^\dagger - \tilde{b}^\dagger\tilde{b}^2 = [2]_{q^{(2N+1)}}\tilde{b}. \tag{18}$$

The structure relations (14) of $\mathrm{osp}_q(1, 2)$ can now be realized by setting

$$E = \tilde{b}^\dagger \qquad F = \tilde{b} \qquad H = 2N. \tag{19}$$

For each p, this defines an irreducible representation of $\mathrm{osp}_q(1, 2)$ on $\mathscr{H}^{(p)}$.

We now turn to fermi-like oscillators. It is well known [15] that in the classical (undeformed) case the parafermi annihilation and creation operators form realizations of the classical algebra su(2). It is hence natural to introduce q-analogues of these parafermi operators using the representations of the quantum algebra $\mathrm{su}_q(2)$. Recall that $\mathrm{su}_q(2)$ is generated by three elements J_0, J_+ and J_- obeying the following commutation relations

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0]_q. \tag{20}$$

It possesses irreducible representations labelled by the index $j = 0, \frac{1}{2}, 1, \ldots$ and defined by [21, 22]

$$\begin{aligned} J_0|j, m\rangle &= m|j, m\rangle \\ J_\pm|j, m\rangle &= ([j \mp m]_q[j \pm m + 1]_q)^{1/2}|j, m \pm 1\rangle \end{aligned} \tag{21}$$

with $-j \leq m \leq j$.

Let us now respectively identify the operators J_- and J_+ of $\mathrm{su}_q(2)$ with the annihilation and creation operators ψ and $\psi^\dagger$ of a q-deformed parafermi oscillator. Recast the spectrum of $J_0 \equiv N$ in the form

$$N_n = n - \frac{p}{2} \qquad n = 0, 1, \ldots, p \tag{22}$$

with the order p equal to $2j$, and identify thereby the set of basis vectors $\{|n\rangle\}$ of the fermionic Fock space $\mathscr{H}^{(p)}$ with the set $\{|j, m\rangle\}$. As in the bosonic case, one still has

$$[N, \psi] = -\psi \qquad [N, \psi^\dagger] = \psi^\dagger. \tag{23}$$

The non-vanishing matrix elements of ψ and $\psi^\dagger$ in $\mathscr{H}^{(p)}$ are then straightforwardly inferred from (21) and one gets

$$\psi_{n,n+1} = \psi^\dagger_{n+1,n} = ([n+1]_q[p-n]_q)^{1/2}. \tag{24}$$

Equations (23) and (24) define a q-analogue of the parafermi oscillator. It is easy to see that in the limit $q \to 1$, they reduce to the standard parafermi relations [15], with $N \to \frac{1}{2}(\psi^\dagger\psi - \psi\psi^\dagger)$.

The commutation relations that the first- and second-order operators satisfy are easily obtained. When $p = 1$, we simply have

$$\psi\psi^\dagger + \psi^\dagger\psi = 1 \qquad \psi^2 = (\psi^\dagger)^2 = 0. \tag{25}$$

It is sometimes practical to write the first of these relations as follows [4, 5, 9]

$$\psi\psi^\dagger + q\psi^\dagger\psi = q^{N+1/2}. \tag{26}$$

In this form it parallels (6) and since $N = \pm\frac{1}{2}$ for $p = 1$, it is equivalent to the former. In the case $p = 2$, we find

$$\psi^3 = 0 \qquad \psi\psi^\dagger\psi = [2]_q\psi \tag{27a}$$

$$\psi^2\psi^\dagger + \psi^\dagger\psi^2 = [2]_q\psi \tag{27b}$$

plus the Hermitian conjugate relations. When $q \to 1$, these trivially reduce to the undeformed parafermi commutation relations of second order.

Let us mention that the 'metaplectic' representation of the quantum superalgebra $\mathrm{osp}_q(3, 2)$ has already been constructed [10] using the $p = 1$ annihilation and creation operators of one q-bose oscillator and one q-fermi oscillator. The bose and fermi operators were taken to mutually commute. The generalization to higher-order cases does not seem immediate. Also, it would be of interest to extend our study to situations involving more than one q-oscillator of order greater than 1. We hope to return to this question.

We thank V P Spiridonov and T D Palev for discussions. This work was initiated thanks to the hospitality and support extended to one of us (LV) by the Istituto Nazionale di Fisica Nucleare, Sezione di Trieste. The work of LV is also funded in part by the UCLA Department of Physics, the National Sciences and Engineering Research Council (NSERC) of Canada and the Fonds FCAR of the Quebec Ministry of Education.

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